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Journal of Approximation Theory 124 (2003) 232–241
JOURNAL OF Approximation Theory
http://www.elsevier.com/locate/jat

Approximation of the rth differential operator by means of linear shape preserving operators of finite rank ☆

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Received 15 August 2002; accepted in revised form 16 September 2003

Communicated by Dany Leviatan

Abstract

In this paper, we present some results estimating the order of approximation of the rth derivative of a function by means of linear operators under different assumptions related to shape preserving properties. We give an example of an operator with the best order of approximation.

Keywords: Korovkin-type results; Shape preserving property; Order of approximation

1. Introduction

In various applications it is necessary to approximate a function preserving such properties as monotonicity, convexity, concavity, etc. The so-called shape preserving operators could serve as a tool for such an approximation.

As this paper will show, if an operator of finite rank has some shape preserving properties, then the order of approximation by the derivatives of the operator is low.

Let $X = [0, 1]$. Denote by $\mathbb{C}^k(X)$, $k \geqslant 0$, the space of all real-valued and k-times continuously differentiable functions on X. Let D^i be the ith differential operator; $|| \cdot ||$ denotes the sup-norm in $\mathbb{C}(X) = \mathbb{C}^0(X)$, $||f|| = \sup_{x \in X} |f(x)|$.

☆ This work is supported by the program "Chief Scientific School of Russian Federation" (project NS-1295.2003.1).
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0021-9045/$ - see front matter
doi:10.1016/j.jat.2003.09.011

Denote by Π_k the subspace of $\mathbb{C}(X)$ spanned by $\{e_0, e_1, \dots, e_k\}$, where $e_i(x) = x^i$, i.e. $\Pi_k = \langle e_0, \dots, e_k \rangle$.

Let $\sigma = (\sigma_i)_{i \geqslant 0}$ be a sequence with $\sigma_i \in \{-1, 0, 1\}$, and let h, k be two integers with $0 \leqslant h \leqslant k$ and $\sigma_h \cdot \sigma_k \neq 0$. Let

$$C_{h,k}(\sigma) = \{f \in \mathbb{C}^k(X) : \sigma_i \cdot D^i f \geqslant 0, \ i = h, \dots, k\}.$$

Let $\Gamma = \{i : h \leqslant i < k, \ \sigma_i \neq 0, \ \sigma_{i+1} = 0, \ \sigma_i \cdot \sigma_{i+2} \neq -1\}$.

When $\Gamma = \emptyset$, then following [7] we call $C_{h,k}(\sigma)$ a cone of type I. When $\Gamma \neq \emptyset$ then we call $C_{h,k}(\sigma)$ a cone of type II.

A linear operator mapping $\mathbb{C}(X)$ into a linear space of finite dimension $n+1$ is called an *operator of finite rank* $n+1$.

In the theory of approximation the well-known result of Korovkin [4] gives conditions that guarantee the convergence of sequences of linear positive operators to the identity operator.

Note that interest in conservative and shape preserving approximation has increased in last years. In particular, Gonska [1], Knoop and Pottinger [2], Muñoz-Delgado and Cárdenas-Morales [6] presented some quantitative Korovkin-type estimates on shape preserving approximation. The next generalizations of Korovkin theorem [4] were presented in [7].

Proposition 1. *Let* $C_{h,k}(\sigma)$ *be a cone of type* I *or* II *and let* $\{L_n\}_{n \geqslant 1}$, $L_n : \mathbb{C}^k(X) \to \mathbb{C}^k(X)$, *be a sequence of linear operators.*

If

$$L_n(C_{h,k}(\sigma)) \subset C_{k,k}(\sigma) \tag{1}$$

and

$$\lim_{n \to \infty} \|D^k L_n e_j - D^k e_j\| = 0 \quad \text{for every } j = h, \dots, k+2,$$

then

$$\lim_{n \to \infty} \|D^k L_n f - D^k f\| = 0 \quad \text{for all } f \in \mathbb{C}^k(X).$$

Proposition 2. *Let* $C_{h,k}(\sigma)$ *be a cone of type* II *and* $r \in \Gamma$. *Let* $\{L_n\}_{n \geqslant 1}$, $L_n : \mathbb{C}^k(X) \to \mathbb{C}^r(X)$, *be a sequence of linear operators.*

If

$$L_n(C_{h,k}(\sigma)) \subset C_{r,r}(\sigma) \tag{2}$$

and

$$\lim_{n \to \infty} \|D^r L_n e_j - D^r e_j\| = 0 \quad \text{for every } j = h, \dots, k, \tag{3}$$

then

$$\lim_{n \to \infty} \|D^r L_n f - D^r f\| = 0 \quad \text{for all } f \in \mathbb{C}^k(X).$$

One of the shortcomings of linear positive operators is their slow convergence. It was shown by Korovkin [5] that the order of approximation by positive linear polynomial operators of degree n cannot be better than n^{-2} in $\mathbb{C}(X)$ even for the functions e_0, e_1, e_2. Moreover, Videnskii [12] has shown that the result of [5] does not depend on the properties of the polynomials but rather on the limitation of dimension.

Extending the results of Korovkin [5] and Videnskii [12] it was shown in [8] that if in the approximation process given in Proposition 1, the operators L_n are assumed to be of finite rank $n+1$, then the order of convergence of $D^k L_n f$ to $D^k f$ cannot be better than n^{-2} even for functions $e_k,\ e_{k+1},\ e_{k+2}$.

In this paper we present similar results for sequences of linear operators L_n of finite rank $n+1$ under the conditions of Proposition 2. Interestingly, we show that an order of approximation better than n^{-2}, may occur. Specifically, we first show that it cannot be better than $n^{-(k-r)}$, and then we present an example with this optimal order.

Note that particular case $k = r+2$ was considered in [9].

2. Main results

Theorem 3. *Let $C_{h,k}(\sigma)$ be a cone of type* II *and $r \in \Gamma$. Let $L_n : \mathbb{C}^k(X) \to \mathbb{C}^r(X)$ be a linear operator of finite rank $n+1$, such that*

(1) $L_n(C_{h,k}(\sigma)) \subset C_{r,r}(\sigma)$;
(2) $D^r L_n e_r = D^r e_r$;
(3) *if* $r > 0$, *then* $L_n(\Pi_{r-1}) \subset \Pi_{r-1}$.

Then

$$\sum_{p=r+1}^{k} \frac{1}{p!} ||D^r L_n e_p - D^r e_p|| \geqslant \frac{1}{\tau (n+1)^{k-r}}, \tag{4}$$

where the constant $\tau > 1$ does not depend on n.

Proof. The proof of the theorem mostly uses the ideas and the technique of [10–12].

It is clear that $k \geqslant r+2$. We assume $\sigma_r = 1$ (for $\sigma_r = -1$ the theorem can be proved similarly).

Let $\{u_j\}_{j=0}^n$ be a system generating the linear space $\{L_n f : f \in \mathbb{C}^k(X)\}$. Consider the matrix

$$A = ||D^r u_j(z_i)||_{j=0,\dots,n;\ i=0,\dots,n+1},$$

where $z_i = \frac{i}{n+1}$, $i = 0, \dots, n+1$.

The rank of A is not equal to zero, rank $A \neq 0$. Indeed, if rank $A = 0$, then $D^r L_n f(z_j) = \sum_{i=0}^n a_i(f) D^r u_i(z_j) = 0, j = 0, \dots, n$, for every $f \in \mathbb{C}^k(X)$, which contradicts hypothesis (2).

Take a non-trivial vector $\delta = \{\delta_i\}_{i=0}^{n+1}$ such that

$$\sum_{i=0}^{n+1} |\delta_i| = 1, \quad \sum_{i=0}^{n+1} \delta_i D^r u_j(z_i) = 0, \quad j = 0, \dots, n.$$

It follows from [3, pp. 82–96] that there exists a constant $c \geqslant 1$, independent of n, and a function $g \in \mathbb{C}^k(X)$ such that

(1) $D^r g(z_i) = \operatorname{sgn} \delta_i$, $i = 0, \dots, n+1$,
(2) $D^i g(0) = 0$, $i = 0, \dots, r-1$,
(3) $||D^i g|| \leqslant c^{i-r}(n+1)^{i-r}$, $i = r, \dots, k$.

As $D^r L_n g$ belongs to the linear space spanned by $\{D^r u_j\}_{j=0}^n$, we get

$$\sum_{i=0}^{n+1} \delta_i D^r L_n g(z_i) = 0.$$

Then

$$\begin{aligned} 1 = \sum_{i=0}^{n+1} |\delta_i| &= \sum_{i=0}^{n+1} \delta_i D^r g(z_i) = \sum_{i=0}^{n+1} \delta_i (D^r g(z_i) - D^r L_n g(z_i)) \\ &\leqslant \sum_{i=0}^{n+1} |\delta_i| |D^r L_n g(z_i) - D^r g(z_i)| \leqslant ||D^r L_n g - D^r g||. \end{aligned} \tag{5}$$

Take $x \in X$ and define two functions $q_{j,x}$, $j = 1, 2$, by

$$q_{j,x}(t) = \sum_{\substack{i=h \\ i \neq r, r+1}}^{k} \frac{1}{i!} m_i (t-x)^i + (-1)^{j+1} D^{r+1} g(x) \frac{1}{(r+1)!} (t-x)^{r+1}, \tag{6}$$

where

$$m_i = \begin{cases} \sigma_i \cdot \left(||D^i g|| + \sum\limits_{\substack{p=i+1 \\ p \neq r}}^{k} ||D^p g|| \cdot \sum\limits_{j=1}^{p} \dfrac{1}{j!} + \dfrac{1}{(r-i)!} ||D^r g|| \right) & \text{if } h < r \text{ and} \\ \quad i = h, \dots, r-1, \\ \sigma_i \cdot \left(||D^i g|| + \sum\limits_{p=i+1}^{k} ||D^p g|| \cdot \sum\limits_{j=1}^{p} \dfrac{1}{j!} \right) \quad \text{if } r < k-2 \text{ and} \\ \quad i = r+2, \dots, k-1, \\ \sigma_k \cdot ||D^k g|| \quad \text{if } i = k. \end{cases}$$

Then

$$\gamma_{j,x} \overset{\text{def}}{=} q_{j,x} + (-1)^j \left(g - \frac{1}{r!} e_r D^r g(x) \right) \in C_{h,k}(\sigma), \quad j = 1, 2. \tag{7}$$

Indeed, since $\gamma_{j,x}$, $j = 1, 2$, satisfy the following properties:

(1) $D^r \gamma_{j,x}(x) = D^r q_{j,x}(x) = 0$,

(2) $D^{r+1}\gamma_{j,x}(x) = D^{r+1}q_{j,x}(x) + (-1)^j D^{r+1}g(x) = 0,$
(3) $D^{r+2}\gamma_{j,x} = D^{r+2}q_{j,x} + (-1)^j D^{r+2}g \geqslant 0,$

we have $D^r\gamma_{j,x} \geqslant 0$.

Finally, it is easy to check that

$$\sigma_i D^i \gamma_{j,x} \geqslant 0, \quad i = h, \dots, k, \ \ i \neq r.$$

Now it follows from hypothesis (1) and (7) that $L_n\gamma_{j,x} \in C_{r,r}(\sigma)$, $j = 1, 2$. Thus,

$$D^r L_n\left(g - \frac{1}{r!} e_r D^r g(x)\right)(x) \leqslant D^r L_n q_{1,x}(x)$$

and

$$-D^r L_n\left(g - \frac{1}{r!} e_r D^r g(x)\right)(x) \leqslant D^r L_n q_{2,x}(x).$$

Consequently,

$$\begin{aligned} &|D^r L_n g(x) - D^r g(x)| \\ &\quad = \left| D^r L_n\left(g - \frac{1}{r!} e_r D^r g(x)\right)(x) + \frac{1}{r!} D^r g(x) \cdot (D^r L_n e_r - D^r e_r)(x) \right| \\ &\quad \leqslant \max\{|D^r L_n q_{1,x}(x)|, |D^r L_n q_{2,x}(x)|\}, \end{aligned} \tag{8}$$

where we have used hypothesis (2).

On the other hand, it follows from

$$\begin{aligned} D^r L_n((\cdot - x)^i)(x) &= D^r L_n\left(\sum_{s=0}^{i} (-1)^{i-s} C_i^s e_s x^{i-s}\right)(x) \\ &= \sum_{s=0}^{i} (-1)^{i-s} C_i^s x^{i-s} (D^r L_n e_s - D^r e_s)(x) \\ &\quad + \sum_{s=0}^{i} (-1)^{i-s} C_i^s x^{i-s} D^r e_s(x) \end{aligned}$$

and

$$\sum_{s=0}^{i} (-1)^{i-s} C_i^s x^{i-s} D^r e_s(x) = i! x^{i-r} \sum_{s=r}^{i} (-1)^{i-s} \frac{1}{(i-s)!(s-r)!} = 0,$$

that

$$D^r L_n((\cdot - x)^i)(x) = \begin{cases} \sum\limits_{s=r+1}^{i} (-1)^{i-s} C_i^s (D^r L_n e_s - D^r e_s)(x), & i > r, \\ 0, \quad 0 \leqslant i \leqslant r. \end{cases} \tag{9}$$

It follows from (6), (8) and (9) that

$$
\begin{aligned}
&|D^r L_n g(x) - D^r g(x)| \\
&\leqslant \sum_{i=r+2}^{k} \left(\frac{1}{i!} \cdot |m_i| \cdot \sum_{s=r+1}^{i} C_i^s |D^r L_n e_s(x) - D^r e_s(x)| \right) \\
&\quad + \frac{1}{(r+1)!} \cdot |D^{r+1} g(x)| \cdot |D^r L_n e_{r+1}(x) - D^r e_{r+1}(x)| \\
&= \sum_{p=r+1}^{k} \left(\frac{1}{p!} \cdot |D^r L_n e_p(x) - D^r e_p(x)| \cdot \sum_{\substack{j=p \\ j \neq r+1}}^{k} \frac{|m_j|}{(j-p)!} \right) \\
&\quad + \frac{1}{(r+1)!} \cdot |D^{r+1} g(x)| \cdot |D^r L_n e_{r+1}(x) - D^r e_{r+1}(x)|. \qquad (10)
\end{aligned}
$$

It follows from $|m_p| \leqslant 2\sum_{l=p}^{k} c^{l-r}(n+1)^{l-r}$, $p = r+2, \ldots, k$, and (5), (10) that

$$
\begin{aligned}
1 &\leqslant 2 \sum_{p=r+1}^{k} \left(\frac{1}{p!} ||D^r L_n e_p - D^r e_p|| \cdot \sum_{j=p}^{k} \left(\frac{1}{(j-p)!} \cdot \sum_{l=j}^{k} c^{l-r}(n+1)^{l-r} \right) \right) \\
&\leqslant 2c^{k-r}(n+1)^{k-r} \sum_{p=r+1}^{k} \frac{1}{p!} ||D^r L_n e_p - D^r e_p|| \sum_{j=p}^{k} \frac{k-j+1}{(j-p)!} \\
&\leqslant 2c^{k-r}(n+1)^{k-r} \sum_{p=r+1}^{k} \frac{1}{p!} ||D^r L_n e_p - D^r e_p|| (k-p+1)e \\
&\leqslant 2e(k-r)c^{k-r}(n+1)^{k-r} \sum_{p=r+1}^{k} \frac{1}{p!} ||D^r L_n e_p - D^r e_p||, \qquad (11)
\end{aligned}
$$

where we have used that $\sum_{j=p}^{k} \frac{1}{(j-p)!} \leqslant e = 2,718\ldots$. Finally (4) follows directly from (11) taking $\tau = 2e \cdot (k-r)c^{k-r}$. $\square$

3. Auxiliary lemmas

Let us consider some lemmas first.

Denote by $[y_0, y_1, \ldots, y_p; f]$ the pth order divided difference of the function f at the knots $y_0 < y_1 < \cdots < y_p$.

Denote by $L_p f(x; y_0, y_1, \ldots, y_p)$ the Newton interpolation polynomial of the function f at the knots $y_0, y_1, \ldots, y_p$:

$$
L_p f(x; y_0, y_1, \ldots, y_p) = \sum_{j=0}^{p} [y_0, \ldots, y_j; f] \cdot \prod_{i=-1}^{j-1} (x - y_i), \quad x - y_{-1} \stackrel{\text{def}}{=} 1. \qquad (12)
$$

Lemma 4. *Let* $p \in \mathbb{N}$ *and* $0 \leqslant x_0 < x_1 < \cdots < x_p \leqslant 1$. *Denote* $x_{-1} = -\infty$, $x_{p+1} = +\infty$. *Let* $f \in C_{0,p+1}(\sigma)$.

(1) *If* $\sigma_0 \sigma_{p+1} > 0$ *then*

$$\sigma_0 L_p f(x; x_0, \ldots, x_p) \geqslant 0 \tag{13}$$

for all $x \in \bigcup_{k=0}^{[p/2]} [x_{p-(2k+1)}, x_{p-2k}]$.

(2) *If* $\sigma_0 \sigma_{p+1} < 0$ *then* (13) *holds for all* $x \in \bigcup_{k=-1}^{[(p-1)/2]} [x_{p-(2k+2)}, x_{p-(2k+1)}]$.

Proof. Let $x \in [x_{l-1}, x_l]$, $l = 0, \ldots, p+1$. It follows from $\sigma_{p+1} D^{p+1} f \geqslant 0$ that $\sigma_{p+1}[x_0, \ldots, x_{l-1}, x, x_l, \ldots, x_p; f] \geqslant 0$, i.e. $\sigma_{p+1} \Delta_p f(x; x_0, \ldots, x_p) \geqslant 0$, where

$$\Delta_p f(x; x_0, \ldots, x_p) \stackrel{\text{def}}{=} (-1)^l \begin{vmatrix} 1 & x & \cdots & x^p & f(x) \\ 1 & x_0 & \cdots & x_0^p & f(x_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_p & \cdots & x_p^p & f(x_p) \end{vmatrix}.$$

It follows from the equality

$$\Delta_p f(x; x_0, \ldots, x_p) = (-1)^{p+l} (L_p f(x; x_0, \ldots, x_p) - f(x)) \prod_{0 \leqslant i < j \leqslant p} (x_j - x_i),$$

that $\sigma_{p+1}(-1)^{p+l} L_p f(x; x_0, \ldots, x_p) \geqslant \sigma_{p+1}(-1)^{p+l} f(x)$. Since $\sigma_0 f \geqslant 0$ inequality (13) holds for appropriate x. □

We need the following well-known properties of the Newton interpolation polynomial.

Lemma 5. *Let* $L_p f(x; y_0, \ldots, y_p)$ *be the Newton interpolation polynomial of the function* f *at the knots* $y_0, \ldots, y_p$. *Then*

(1) $L_p e_i(x; y_0, \ldots, y_p) = e_i(x)$, $i = 0, \ldots, p$,
(2) $L_p e_{p+1}(x; y_0, \ldots, y_p) = e_{p+1}(x) - \prod_{j=0}^{p} (x - y_j)$.

4. Example

Let $p, k, n \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$ and $2 < k - r < n$.

For $f \in \mathbb{C}^r(X)$ denote by $L_{r,p} f(x; y_0, \ldots, y_p)$ the polynomial of degree $r + p$ in the variable x uniquely determined by

(1) $D^r L_{r,p} f(x; y_0, \ldots, y_p) = L_p(D^r f)(x; y_0, \ldots, y_p)$,
(2) $D^i L_{r,p} f(0; y_0, \ldots, y_p) = 0$, $i = 0, \ldots, r-1$, if $r > 0$.

Let $C_{h,k}(\sigma)$ be a cone of type II with $r\in\Gamma$.

Denote $x_{i,n}=i/n,\ i=0,\dots,n$.

Let $\Lambda^{[\sigma]}_{r,k,n}:\mathbb{C}^r(X)\to\mathbb{C}^r(X)$ be a linear operator defined by

$$\Lambda^{[\sigma]}_{r,k,n}f(x)=\begin{cases}\sum_{l=0}^{r-1}\frac{1}{l!}x^l(D^lf(x_{0,n})-D^lA_{1,r,k,n}f(x_{0,n}))\\ \quad+A_{1,r,k,n}f(x),\quad x\in[x_{0,n},x_{1,n}],\\ \sum_{l=0}^{r-1}\frac{1}{l!}(x-x_{j-1,n})^l\Big(D^l\Lambda^{[\sigma]}_{r,k,n}f(x_{j-1,n})-D^lA_{j,r,k,n}f(x_{j-1,n})\Big)\\ \quad+A_{j,r,k,n}f(x),\quad x\in(x_{j-1,n},x_{j,n}],\ j=2,\dots,n,\end{cases}\tag{14}$$

where

(1) if $k-r$ is an odd number and $\sigma_r\sigma_k>0$, then

$$A_{j,r,k,n}f(x)=\begin{cases}L_{r,k-r-1}f(x;x_{1,n},\dots,x_{k-r,n}),\quad j=2i+1,\\ \quad i=0,\dots,(k-r-1)/2,\\ L_{r,k-r-1}f(x;x_{0,n},\dots,x_{k-r-1,n}),\quad j=2i+2,\\ \quad i=0,\dots,(k-r-3)/2,\\ L_{r,k-r-1}f(x;x_{j-(k-r-1),n},\dots,x_{j,n}),\quad j=k-r+1,\dots,n,\end{cases}\tag{15}$$

(2) if $k-r$ is an odd number and $\sigma_r\sigma_k<0$, then

$$A_{j,r,k,n}f(x)=\begin{cases}L_{r,k-r-1}f(x;x_{j-1,n},\dots,x_{j+k-r-2,n}),\quad j=1,\dots,n-(k-r),\\ L_{r,k-r-1}f(x;x_{n-(k-r),n},\dots,x_{n-1,n}),\quad j=n-2i,\\ \quad i=0,\dots,(k-r-1)/2,\\ L_{r,k-r-1}f(x;x_{n-(k-r-1),n},\dots,x_{n,n}),\quad j=n-2i-1,\\ \quad i=0,\dots,(k-r-3)/2,\end{cases}\tag{16}$$

(3) if $k-r$ is an even number and $\sigma_r\sigma_k>0$, then

$$A_{j,r,k,n}f(x)=\begin{cases}L_{r,k-r-1}f(x;x_{0,n},\dots,x_{k-r-1,n}),\quad j=2i+1,\\ \quad i=0,\dots,(k-r-2)/2,\\ L_{r,k-r-1}f(x;x_{1,n},\dots,x_{k-r,n}),\quad j=2i+2,\\ \quad i=0,\dots,(k-r-2)/2,\\ L_{r,k-r-1}f(x;x_{j-(k-r-1),n},\dots,x_{j,n}),\quad j=k-r+1,\dots,n,\end{cases}\tag{17}$$

(4) if $k-r$ is an even number and $\sigma_r\sigma_k<0$, then

$$A_{j,r,k,n}f(x) = \begin{cases} L_{r,k-r-1}f(x; x_{1,n}, \dots, x_{k-r,n}), & j=1, \\ L_{r,k-r-1}f(x; x_{j-2,n}, \dots, x_{j+k-r-3,n}), & j=2,\dots, \\ \quad n-(k-r-1), & \\ L_{r,k-r-1}f(x; x_{n-(k-r),n}, \dots, x_{n-1,n}), & j=n-2i, \\ \quad i=0,\dots,(k-r-2)/2, & \\ L_{r,k-r-1}f(x; x_{n-(k-r-1),n}, \dots, x_{n,n}), & j=n-2i-1, \\ \quad i=0,\dots,(k-r-4)/2. & \end{cases} \tag{18}$$

It is obvious that $\Lambda^{[\sigma]}_{r,k,n}$ is an operator of finite rank $n+1$.

Theorem 6. *Let* $\Lambda^{[\sigma]}_{r,k,n}: \mathbb{C}^k(X) \to \mathbb{C}^r(X)$ *be a linear operator defined by* (14)–(18). *Then*

(1) $\Lambda^{[\sigma]}_{r,k,n}(C_{r,k}(\sigma)) \subset C_{r,r}(\sigma)$;
(2) $D^r\Lambda^{[\sigma]}_{r,k,n}e_i = D^r e_i$, $i=0,\dots,k-1$;
(3) $\|D^r\Lambda^{[\sigma]}_{r,k,n}e_k - D^r e_k\| = \frac{k!}{n^{k-r}}$, *if*
   (a) $k-r$ *is an odd number*,
   (b) $k-r$ *is an even number and* $\sigma_r\sigma_k < 0$;
(4) $\|D^r\Lambda^{[\sigma]}_{r,k,n}e_k - D^r e_k\| = \frac{1}{n^{k-r}}\frac{k!}{(k-r)!}\sup_{x\in X}\prod_{j=0}^{k-r-1}|x-j|$, *if* $k-r$ *is an even number and* $\sigma_r\sigma_k > 0$;
(5) *for every* $f \in \mathbb{C}^k(X)$, $\lim_{n\to\infty}\|D^r\Lambda^{[\sigma]}_{r,k,n}f - D^r f\| = 0$.

Proof. Condition (1) follows from Lemma 4. Apply it to $D^r f$ with $f \in C_{r,k}(\sigma)$. Notice that $D^r f \in C_{0,k-r}(\sigma^*)$ with $\sigma^*_i = \sigma_{r+i}$. Conditions (2)–(4) follow from Lemma 5. Finally (5) is a direct consequence of (1)–(4) and Proposition 2. □

5. Conclusion

The following statement is a direct consequence of Theorems 3 and 6:

Let h, r, k be three integers with $0 \leqslant h \leqslant r \leqslant k-2$. Let $C(h,r,k) = \{C_{h,k}(\sigma): \sigma_h \cdot \sigma_k \neq 0,\ \sigma_r \neq 0,\ \sigma_{r+1} = 0,\ \sigma_r\sigma_{r+2} \neq -1\}$. Denote by $\mathscr{L}_n(h,r,k)$ a set of linear operators $L_n: \mathbb{C}^k(X) \to \mathbb{C}^r(X)$ of finite rank $n+1$, such that

(1) there is a cone $C_{h,k}(\sigma) \in C(h,r,k)$ such that $L_n(C_{h,k}(\sigma)) \subset C_{r,r}(\sigma)$;
(2) $D^r L_n e_r = D^r e_r$;
(3) if $r > 0$, then $L_n(\Pi_{r-1}) \subset \Pi_{r-1}$.

Then

$$\frac{1}{\tau(n+1)^{k-r}} \leqslant \inf_{L_n \in \mathscr{L}_n(h,r,k)} \sum_{p=r+1}^{k} \frac{1}{p!}\|D^r L_n e_p - D^r e_p\| \leqslant \frac{1}{n^{k-r}},$$

where the constant $\tau > 1$ does not depend on n.

Acknowledgments

I thank Professor A.L. Lukashov for his helpful suggestions. I express my gratitude to the referee for the careful perusal, finding inaccuracies and valuable remarks.

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